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THE PLANE PROBLEM FOR A NONHOMOGENEOUS ELASTIC CIRCULAR DOMAIN

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We consider the problem of a layered-nonhomogeneous body such that the elastic constants of the isotropic medium depend exponentially on the Cartesian coordinates. Poisson's ratio is assumed to be constant. Making use of the Airy stress function and of expansions with respect to a small parameter, we construct the solution of the first fundamental boundary value problem of the plane elasticity theory for a circular domain when on the circumference we are given the radial stresses as continuous functions of bounded variation of the polar angle.

This problem has been investigated in [1] by the method of successive approximations with the aid of the complex representations given by the authors of [2 – 4]. The convergence of the successive approximation process has not been proved.

In problems with nonhomogeneous elasticity the complex representations present no advantage, therefore we will apply the Airy function method in that version in which it was used in [5], in conjunction with expansions with respect to a small parameter, characterizing the nonhomogeneity. We give a recursion process in order to compute the functions which are the coefficients of the power series and we prove the convergence of this series. An example is given.

1. In the plane problem for a nonhomogeneous isotropic medium with a constant Poisson's ratio ν, with a nonhomogeneity in the form of an exponential function and in the absence of body forces and temperature stresses, the equation for the Airy function F in polar coordinates has the form

$$\Delta\Delta F + 2qPF + q^2\Delta F = \frac{q^2}{1-\nu} QF \tag{1.1}$$

$$PF = \left(\cos\varphi \frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi}\right)\Delta F, \qquad \Delta = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

$$QF = \left[\sin^2\varphi \frac{\partial^2 F}{\partial r^2} + \cos^2\varphi\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\varphi^2}\right) + \sin 2\varphi \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\varphi}\right)\right]$$

where q is a small parameter characterizing the exponential dependence $\mu = \mu_0 e^{qx}$ of the modulus of elasticity of the medium. Here x is a dimensionless abscissa related to the radius of the circle $r = a$. At the boundary of the domain the stress function $F(r, \varphi)$ satisfies the boundary condition

$$\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\varphi^2}\Big|_{\substack{r=1\\ 0\leqslant\varphi<2\pi}} = \sigma(\varphi) \tag{1.2}$$

In addition, $F(r, \varphi)$ is bounded for $r = 0$. The solution is sought in the class of analytic functions in the form of a power series in the small parameter q

$$F(r, \varphi) = F_0(r, \varphi) + qF_1(r, \varphi) + q^2F_2(r, \varphi) + \ldots + q^nF_n(r, \varphi) + \ldots \tag{1.3}$$

The first term of the series (1.3) corresponds to a body subject to the same load but assumed to be homogeneous. The subsequent terms introduce the corrections due to the nonhomogeneity. As a result of inserting (1.3) into Eq. (1.1) and taking into account the boundary conditions (1.2), we obtain the following system of boundary value problems for the coefficients of the series (1.3) with the corresponding boundary conditions:

1) $\Delta\Delta F_0 = 0$, where F_0 satisfies the boundary condition

$$\frac{1}{r}\frac{\partial F_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F_0}{\partial \varphi^2}\Big|_{\substack{r=1\\ 0\leqslant\varphi<2\pi}} = \sigma(\varphi) \tag{1.4}$$

and $F_0(r, \varphi)$ is bounded for $r = 0$;

2) $\Delta\Delta F_1 = -2PF_0$

$$\Delta\Delta F_n = -2PF_{n-1} - \Delta F_{n-2} + \frac{1}{1-\nu}QF_{n-2}, \qquad n = 2, 3, \ldots \tag{1.5}$$

where all $F_n(r, \varphi)$ $(n = 1, 2, 3, \ldots)$ satisfy the zero condition at the boundary of the domain, i.e.

$$\frac{1}{r}\frac{\partial F_n}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F_n}{\partial \varphi^2}\Big|_{\substack{r=1\\ 0\leqslant\varphi<2\pi}} = 0 \tag{1.6}$$

and, in addition $F_n(r, \varphi)$ are bounded for $r = 0$.

Making use of the solution of the first fundamental boundary value problem for a homogeneous medium, given in [4], Sect. 54, we have

$$F_0(r, \varphi) = \frac{\alpha_0}{4}r^2 + \sum_{m=2}^{\infty}\left[\frac{r^{m+2}}{2(m+1)} - \frac{r^m}{2(m-1)}\right](\alpha_m \cos m\varphi + \beta_m \sin m\varphi) \tag{1.7}$$

where α_0, α_m, β_m are the coefficients in the expansion of $\sigma(\varphi)$ in a Fourier series

$$\sigma(\varphi) = \frac{\alpha_0}{2} + \sum_{m=1}^{\infty}(\alpha_m \cos m\varphi + \beta_m \sin m\varphi) \tag{1.8}$$

The remaining coefficients of the series (1.3) have to be determined from the recursion system (1.5) with boundary conditions (1.6). We assume that the coefficients up to the $(n-1)$-st inclusively, have been found. Then the left-hand side of Eq. (1.5) represents a continuous function of both arguments (by virtue of the properties of the operators P, Q, Δ) and can be represented by the trigonometric series

$$\Phi(r, \varphi) = a_0(r) + \sum_{m=1}^{\infty} a_m(r)\cos m\varphi + b_m(r)\sin m\varphi \tag{1.9}$$

We seek the solution of Eq. (1.5) with the boundary conditions (1.6) in the form

$$F_n(r, \varphi) = f_0^n(r) + \sum_{m=1}^{\infty} f_m^n(r)\cos m\varphi + g_m^n(r)\sin m\varphi \tag{1.10}$$

Here and in the sequel, the superscript of the coefficients of the series expansions indicates the index of the coefficient of the series (1.3) for which the expansion is obtained. Inserting (1.10) and (1.9) into Eq. (1.5), we separate the variables and we obtain a system of differential equations for the coefficients $f_m{}^n$ and $g_m{}^n$ $(m = 0, 1, 2, \ldots)$ of the series (1.10)

$$\left(\frac{1}{r}\frac{d}{dr} + \frac{d^2}{dr^2}\right)^2 f_0{}^n(r) = a_0(r)$$
$$\left(\frac{1}{r}\frac{d}{dr} + \frac{d^2}{dr^2} - \frac{1}{r^2}\right)^2 f_1{}^n(r) = a_1(r) \qquad (1.11)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\left(\frac{1}{r}\frac{d}{dr} + \frac{d^2}{dr^2} - \frac{m^2}{r^2}\right)^2 f_m{}^n(r) = a_m(r)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

From the boundary conditions (1.6) we obtain the conditions for $f_m{}^n(r)$ at the boundary of the domain

$$\frac{1}{r}\frac{df_0{}^n}{dr}\bigg|_{r=1} = 0$$
$$\frac{1}{r}\frac{df_m{}^n}{dr} - \frac{m^2}{r^2} f_m{}^n\big|_{r=1} = 0, \qquad m = 1, 2, 3, \ldots \qquad (1.12)$$

We have similar equations and boundary conditions for $g_m{}^n(r)$. In addition, $f_m{}^n(r)$ and $g_m{}^n(r)$ must be bounded for $r = 0$. The Green functions for each of the equations of the system (1.11) with the boundary conditions (1.12) allow us to define

$$F_n(r, \varphi) = \int_0^1 \Gamma_0(r, \xi)\, a_0(\xi)\, d\xi + \sum_{m=1}^{\infty}\left[\cos m\varphi \int_0^1 \Gamma_m(r, \xi)\, a_m(\xi)\, d\xi + \sin m\varphi \int_0^1 \Gamma_m(r, \xi)\, b_m(\xi)\, d\xi\right] \qquad (1.13)$$

where $\Gamma_0(r, \xi)$, $\Gamma_m(r, \xi)$ are the Green functions for the boundary value problems (1.11) with the boundary conditions (1.12)

$$\Gamma_0(r, \xi) = \begin{cases} {}^1/_8\,(2\xi^3 \ln \xi + 2r^2\xi \ln \xi + r^2\xi + \xi - r^2\xi^3 - \xi^3), & 0 \leqslant r \leqslant \xi \leqslant 1 \\ {}^1/_8\,(2\xi^3 \ln r + 2\xi r^2 \ln r - r^2\xi + \xi^3 - r^2\xi^3), & 0 \leqslant \xi \leqslant r \leqslant 1 \end{cases}$$

$$\Gamma_1(r, \xi) = \begin{cases} {}^1/_{16}\,(2r^3\xi^2 - r^3\xi^4 - 2r\xi^2 + 2r\xi^4 - r^3 - 4r\xi^2 \ln \xi), & 0 \leqslant r \leqslant \xi \leqslant 1 \\ {}^1/_{16}\,(2r^3\xi^2 - r^3\xi^4 - r^{-1}\xi^4 + 2r\xi^4 - 2r\xi^2 - 4\xi^2 r \ln r), & 0 \leqslant \xi \leqslant r \leqslant 1 \end{cases}$$

$$\Gamma_m(r, \xi) = \begin{cases} \frac{1}{8}\left[r^m\left(\frac{\xi^{m+1}}{1-m} - \frac{\xi^{m+3}}{m} - \frac{\xi^{3-m}}{m(1-m)}\right) + r^{m+2}\left(\frac{\xi^{m+1}}{m} - \frac{\xi^{m+3}}{1+m} - \frac{\xi^{1-m}}{m(1+m)}\right)\right], & 0 \leqslant r \leqslant \xi \leqslant 1 \\ \frac{1}{8}\left[\xi^{m+1}\left(\frac{r^m}{1-m} + \frac{r^{m+2}}{m} - \frac{r^{2-m}}{m(1-m)}\right) + \xi^{m+3}\left(\frac{r^m}{m} - \frac{r^{m+2}}{m+1} - \frac{r^{-m}}{m(m+1)}\right)\right], & 0 \leqslant \xi \leqslant r \leqslant 1, \quad m = 2, 3, \ldots \end{cases}$$

Thus we obtain

$$F_1(r, \varphi) = -\frac{1}{2^2}\sum_{m=1}^{\infty}\left[\frac{r^{m+4}}{2(m+2)} - \frac{r^{m+2}}{m+2} + \frac{r^m}{2(m+2)}\right](\alpha_{m+1}\cos m\varphi + \beta_{m+1}\sin m\varphi)$$

$$F_2(r,\varphi)=\frac{1}{2^4(1-\nu)}\Big\{\Big[\frac{3-4\nu}{36}\alpha_2 r^6+\Big(\frac{4\nu-1}{12}\alpha_2+\frac{2\nu-1}{8}\alpha_0\Big)r^4+$$
$$\Big(-\frac{1+4\nu}{12}\alpha_2+\frac{1-2\nu}{4}\alpha_0\Big)r^2+\Big(-\frac{3-4\nu}{36}\alpha_2-\frac{1-2\nu}{8}\alpha_0\Big)\Big]+$$
$$\Big[\frac{3-4\nu}{48}\alpha_3 r^7-\frac{1-2\nu}{8}\alpha_3 r^5+\frac{1-4\nu}{16}\alpha_3 r^3+\frac{\nu}{12}\alpha_3 r\Big]\cos\varphi+[\ldots]\sin\varphi+$$
$$\sum_{m=2}^{\infty}\Big[\frac{3-4\nu}{12(m+3)}\alpha_{m+2}r^{m+6}+\Big(\frac{-3m-1+4\nu(m+1)}{4(m+1)(m+3)}\alpha_{m+2}+\frac{\alpha_m}{2(m+1)(m+2)}\Big)\times$$
$$r^{m+4}+\Big(\frac{3m-1-4\nu(m+1)}{4(m+1)(m+2)}\alpha_{m+2}-\frac{\alpha_m}{2(m+1)(m+2)}\Big)r^{m+2}+\qquad(1.15)$$
$$\Big(\frac{-3m+1+4\nu(m+1)}{12(m+1)(m+3)}\alpha_{m+2}+\frac{\alpha_m}{2(m+1)(m+2)}\Big)r^m\Big]\cos m\varphi+[\ldots]\sin m\varphi\Big\}$$

Here and in the sequel, the dots in the brackets at $\sin m\varphi$ $(m=1,2,3,\ldots)$ indicate that in the brackets we have the same polynomial in r as at $\cos m\varphi$ with the substitution of α_m by β_m. As before, α_m and β_m are the coefficients of the Fourier series (1.8).

By mathematical induction we can prove that the nth coefficient of the series (1.3) has the form

$$F_n(r,\varphi)=\frac{(-1)^n}{2^{2n}(1-\nu)^\lambda}\Big[a_0{}^n+\sum_{m=1}^{\infty}a_m{}^n\cos m\varphi+b_m{}^n\sin m\varphi\Big]\quad(1.16)$$

$$\lambda=n/2\qquad\text{for}\quad n\equiv0,\quad n\equiv2\ (\mathrm{mod}\ 4)$$
$$\lambda=(n-1)/2\qquad\text{for}\quad n\equiv1,\quad n\equiv3\ (\mathrm{mod}\ 4)$$

where the coefficients of $\cos m\varphi$ and $\sin m\varphi$ $(m=0,1,2,\ldots)$ are polynomials in r of the form

$$a_m{}^n=\alpha^n_{m,\,m+2n+2}r^{m+2n+2}+\alpha^n_{m,\,m+2n}r^{m+2n}+\ldots+\alpha^n_{m,\,m+2}r^{m+2}+\alpha^n_{m,\,m}r^m$$
$$b_m{}^n=\beta^n_{m,\,m+2n+2}r^{m+2n+2}+\beta^n_{m,\,m+2n}r^{m+2n}+\ldots+\beta^n_{m,\,m+2}r^{m+2}+\beta^n_{m,\,m}r^m$$

In addition, from the Eq. (1.5) it follows that $a_m{}^r$ and $b_m{}^n$ can be expressed in terms of the coefficients of the expansions of F_{n-1} and F_{n-2}. From (1.16) and (1.13) we obtain by straightforward computation the expressions for $a_m{}^n$

$$a_m{}^n=\Big[\frac{2}{n+1}(1-\nu)^\delta\alpha^{n-1}_{m+1,\,m+2n+1}-\frac{1}{n(n+1)}\alpha^{n+2}_{m+2,m+2n}\Big]\times$$
$$[r^{m+2n+2}-(n+1)r^{m+2}+nr^m]+\Big[\frac{2}{n}(1-\nu)^\delta\alpha^{n-1}_{m+1,\,m+2n-1}+$$
$$\frac{2(1-\nu)^\delta}{m+n}\alpha^{n-1}_{m-1,\,m+2n-1}-\frac{1}{(n-1)n}\alpha^{n-2}_{m+2,\,m+2n-2}-\frac{2(1-2\nu)}{n(m+n)}\alpha^{n-2}_{m,\,m+2n-2}\Big]\times$$
$$[r^{m+2n}-nr^{m+2}+(n-1)r^m]+\Big[\frac{2}{n-1}(1-\nu)^\delta\alpha^{n-1}_{m+1,\,m+2n-3}+$$
$$\frac{2(1-\nu)^\delta}{m+n-1}\alpha^{n-1}_{m-1,\,m+2n-3}-\frac{1}{(n-2)(n-1)}\alpha^{n-2}_{m+2,\,m+2n-4}-$$
$$\frac{2(1-2\nu)}{n(m+n)}\alpha^{n-2}_{m,\,m+2n-4}-\frac{1}{(m+n-2)(m+n-1)}\alpha^{n-2}_{m-2,\,m+2n-4}]\times$$
$$[r^{m+2n-2}-(n-1)r^{m+2}+(n-2)r^m]+\ldots$$
$$\Big[(1-\nu)^\delta\alpha^{n-1}_{m+1,\,m+3}+\frac{2}{m+2}(1-\nu)^\delta\alpha^{n-1}_{m-1,\,m+3}-\frac{1}{2}\alpha^{n-2}_{m+2,\,m+2}-$$

$$\frac{1-2\nu}{m+2}\alpha^{n-2}_{m,\,m+2}-\frac{\alpha^{n-2}_{m-2,\,m+2}}{(m+1)(m+2)}\Big][r^{m+4}-2r^{m+2}+r^{m}] \qquad (1.17)$$

$$\delta=\begin{cases}1 & \text{for } n\equiv 0,\ n\equiv 2 \pmod 4\\ 0 & \text{for } n\equiv 1,\ n\equiv 3 \pmod 4\end{cases}$$

The coefficients $b_m{}^n$ for the corresponding values of n are obtained from (1.17) by changing $\alpha^{v}_{p,t}$ into $\beta^{v}_{p,t}$.

2. We define the norm of the function $F(r,\varphi)$ as the sum of the absolute values of the coefficients of its expansion in the trigonometric series (1.10)

$$\|F_n(r,\varphi)\|=\sum_{m=0}^{\infty}(|f_m{}^n(r)|+|g_m{}^n(r)|)$$

According to this definition

$$\|\sigma(\varphi)\|=\frac{|\alpha_0|}{2}+\sum_{m=1}^{\infty}(|\alpha_m|+|\beta_m|)=C$$

From (1.16) we have

$$|f_m{}^n(r)|=\frac{1}{2^{2n}(1-\nu)^{\lambda}}|a_m{}^n|,\qquad |g_m{}^n(r)|=\frac{1}{2^{2n}(1-\nu)^{\lambda}}|b_m{}^n|$$

From (1.14) and (1.15) we obtain at once

$$|\alpha^1_{m,\,m+4}|\leqslant\frac{|\alpha_{m+1}|}{2(m+2)},\qquad |\alpha^2_{m,\,m+6}|<\frac{|\alpha_{m+2}|}{6(m+3)}$$

$$|\alpha^1_{m,\,m+2}|\leqslant\frac{|\alpha_{m+1}|}{m+2},\qquad |\alpha^2_{m,\,m+4}|<\frac{|\alpha_{m}|}{m+3}$$

$$|\alpha^1_{m,\,m}|\leqslant\frac{|\alpha_{m+1}|}{2(m+2)},\qquad |\alpha^2_{m,\,m+2}|<\frac{3}{m+3}|\alpha_m|$$

$$|\alpha^2_{m,\,m}|<\frac{2}{m+3}|\alpha_m|$$

The formulas (1.17) allow us to estimate $|\alpha^3_{m,\,m+p}|$ and $|\alpha^4_{m,\,m+p}|$ for $p=0,2,4,\ldots 10$. We obtain

$$|\alpha^3_{m,\,m+8}|<\frac{1}{3(m+4)}|\alpha_{m+3}|,\qquad |\alpha^4_{m,\,m+10}|<\frac{1}{6(m+5)}|\alpha_{m+4}|$$

$$|\alpha^3_{m,\,m+6}|<\frac{2}{m+4}|\alpha_{m+1}|,\qquad |\alpha^4_{m,\,m+8}|<\frac{3}{m+5}|\alpha_{m+2}|$$

$$|\alpha^3_{m,\,m+4}|<\frac{3^2}{m+4}|\alpha_{m-1}|,\qquad |\alpha^4_{m,\,m+6}|<\frac{15}{m+5}|\alpha_{m}|$$

$$|\alpha^3_{m,\,m+2}|<\frac{3^3}{m+4}|\alpha_{m-1}|,\qquad |\alpha^4_{m,\,m+4}|<\frac{3^4}{m+5}|\alpha_{m-2}|$$

$$|\alpha^3_{m,\,m}|<\frac{2^4}{m+4}|\alpha_{m-1}|,\qquad |\alpha^4_{m,\,m+2}|<\frac{3^5}{m+5}|\alpha_{m-2}|$$

$$|\alpha^4_{m,\,m}|<\frac{2^7}{m+5}|\alpha_{m-2}|$$

By mathematical induction, with the aid of the formulas (1.17), we can prove that

$$|\alpha_{m,\,m+4}^n| < \frac{3^{2(n-2)}}{m+n+1}\,|\alpha_{m-n+2}|$$

and $|\alpha_{m,\,m+4}^n|$ is the greatest among $|\alpha_{m,\,m+p}^n|$ for $p=4, 6, \ldots, 2n+2$. From (1.17) for $n \equiv 1$, $n \equiv 3 \pmod 4$ we have

$$|a_m{}^n| \leqslant 2(n+1)\,|\alpha_{m,\,m+2n+2}^n| + 2n\,|\alpha_{m,\,m+2n}^n| + 2(n-1)\,|\alpha_{m,\,m+2n-2}^n| + \ldots$$

$$2\cdot 2\,|\alpha_{m,\,m+4}^n| < |\alpha_{m,\,m+4}^n|\, n(n+3) < \frac{3^{2(n-2)} n\,(n+3)}{m+n+1}\,|\alpha_{m-n+2}| \qquad (2.1)$$

The estimate obtained holds for $m \geqslant (n-1)$. For $0 \leqslant m \leqslant n-2$

$$|a_m{}^n| < \frac{3^{2(n-2)} n\,(n+3)}{m+n+1}\,|\alpha_0|$$

This follows from the properties of the operators P and Q which occur in the left-hand side of Eq. (1.5). Thus

$$|f_m{}^n(r) < \frac{3^{2(n-2)}(n+3)}{2^{2n}(1-\nu)^{\lambda}}\,|\alpha_{m-n+2}|, \qquad m \geqslant n-1$$

$$|f_m{}^n(r)| < \frac{3^{2(n-2)}(n+3)}{2^{2n}(1-\nu)^{\lambda}}\,|\alpha_0|, \qquad 0 \leqslant m \leqslant n-2$$

Similar estimates are obtained for $|g_m^n(r)|$. From here we obtain the following estimate for $\|F_n\|$:

$$\|F_n(r,\varphi)\| < \frac{3^{2(n-2)}(n+3)(n-2)\,C_1}{2^{2n}(1-\nu)^{\lambda}}$$

$$C_1 = |\alpha_0| + |\beta_1| + \|\sigma(\varphi)\| = |\alpha_0| + |\beta_1| + C$$

Thus, the series (1.3) is majorized by the series

$$\sum_{n=0}^{\infty} \frac{3^{2(n-2)}(n+3)(n-2)\,C_1}{2^{2n}(1-\nu)^{\lambda}}\, q^n$$

which converges absolutely for $|q| < 1/3$.

It should be noted that actually the boundary of the convergence of the series (1.3) is much wider. Moreover, as it will be proved later, even for $|q| = 1.6$ the series converges sufficiently fast.

3. The solution of the first fundamental boundary value problem for a nonhomogeneous circular domain, loaded on the contour by a uniform radial stress $\sigma(\varphi) = p$, is obtained from the solution for the general case.

Thus, from (1.7), (1.14), (1.15) and from the subsequent expressions for F_3, F_4, F_5 we have

$$F_0 = \frac{1}{2}pr^2, \qquad F_1 = 0, \qquad F_2 = -\frac{1-2\nu}{2^6(1-\nu)}\,p(1-r^2)^2$$

$$F_3 = \frac{1-2\nu}{2^6\cdot 3(1-\nu)}\,pr(1-r^2)^2\cos\varphi$$

$$F_4 = -\frac{1-2\nu}{2^8(1-\nu)^2}\,p\left[\left(\frac{3-2\nu}{18}r^6 - \frac{5-2\nu}{12}r^4 + \frac{1}{3}r^2 - \frac{3+2\nu}{36}\right) + \frac{3-4\nu}{24}r^2(1-r^2)^2\cos 2\varphi\right]$$

$$F_5 = \frac{1-2\nu}{2^{10}(1-\nu)^2}\, p \left[\left(\frac{3-2\nu}{18} r^7 + \frac{4\nu-17}{36} r^5 + \frac{4+\nu}{9} r^3 - \frac{5+4\nu}{36} r \right) \cos\varphi + \frac{1-2\nu}{30}(1-r^2)^2 r^3 \cos 3\varphi \right]$$

From here we obtain the expressions for the stress components, corresponding to the first five terms of the series (1.3)

$$\sigma_r{}^{0} = \sigma_\varphi{}^{0} = p, \qquad \tau_{r\varphi}{}^{0} = 0$$

$$\sigma_r{}^{1} = \sigma_\varphi{}^{1} = 0, \qquad \tau_{r\varphi}^{1} = 0$$

$$\sigma_r{}^{2} = \frac{(1-2\nu)\,pq^2}{2^4(1-\nu)}(1-r^2), \qquad \sigma_\varphi{}^{2} = \frac{(1-2\nu)\,pq^2}{2^4(1-\nu)}(1-3r^2), \qquad \tau_{r\varphi} = 0$$

$$\sigma_r{}^{3} = -\frac{(1-2\nu)\,pq^3}{2^4 3(1-\nu)}\, r(1-r^2)\cos\varphi, \qquad \sigma_\varphi{}^{3} = -\frac{(1-2\nu)\,pq^3}{2^4 3(1-\nu)}\, r(3-5r^2)\cos\varphi$$

$$\tau_{r\varphi}^{3} = -\frac{(1-2\nu)\,pq^3}{2^4 3(1-\nu)}\, r(1-r^2)\sin\varphi$$

$$\sigma_r{}^{4} = \frac{(1-2\nu)\,pq^4}{2^{10}3(1-\nu)^2}\left[4(3-2\nu)r^2 - 8 + (3-4\nu)(1+r^2)\cos 2\varphi\right](1-r^2)$$

$$\sigma_\varphi{}^{4} = \frac{(1-2\nu)\,pq^4}{2^{10}3(1-\nu)^2}\left[-20(3-2\nu)r^4 + 12(5-2\nu)r^2 - 8 - (3-4\nu)(1 - 12r^2 + 15r^4)\cos 2\varphi\right], \qquad \tau_{r\varphi}^{4} = \frac{(1-2\nu)\,pq^4}{2^{10}3(1-\nu)^2}\left[(3-4\nu)(-1+6r^2-5r^4)\right]\sin 2\varphi$$

$$\sigma_r{}^{5} = -\frac{(1-2\nu)\,pq^5}{2^{10}(1-\nu)^2}\left\{\left[\frac{3-2\nu}{3} r^5 - \frac{17-4\nu}{9} r^3 + \frac{2(4+\nu)}{9} r\right]\cos\varphi + \frac{(1-2\nu)}{30}(2r^5 - 8r^3 + 6r)\cos 3\varphi\right\}, \qquad \sigma_\varphi{}^{5} = \frac{(1-2\nu)\,pq^5}{2^{10}(1-\nu)^2} \times$$

$$\left\{\left[\frac{7(3-2\nu)}{3} r^5 + \frac{5(4\nu-17)}{9} r^3 + \frac{2(4+\nu)}{3} r\right]\cos\varphi + \frac{1-2\nu}{15}(3r - 20r^3 + 21r^5)\cos 3\varphi\right\}, \qquad \tau_{r\varphi}^{5} = \frac{(1-2\nu)\,pq^5}{2^{10}(1-\nu)^2}\left\{\left[\frac{3-2\nu}{3} r^5 + \frac{4\nu-1}{9} r^3 + \frac{2(4-\nu)}{9} r\right]\sin\varphi + \frac{1-2\nu}{5}(3r^5 - 4r^3 + r)\sin 3\varphi\right\}$$

Table 1.

Stresses	r = 0.0	0.2	0.4	0.6	0.8	1.0
$\sigma_{r_0} = \sigma_{r1}$	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
σ_{r2}	1.09143	1.08777	1.07680	1.05851	1.03291	1.00000
σ_{r3}	1.09143	1.09713	1.09318	1.07723	1.04695	1.00000
σ_{r4}	1.08064	1.08753	1.08676	1.07490	1.04764	1.00000
σ_{r5}	1.08064	1.08839	1.08769	1.07400	1.04703	1.00000
$\sigma_{\varphi 0} = \sigma_{\varphi 1}$	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
$\sigma_{\varphi 2}$	1.09143	1.08046	1.04754	0.99269	0.91589	0.81714
$\sigma_{\varphi 3}$	1.09143	1.10777	1.09045	1.02780	0.90809	0.71962
$\sigma_{\varphi 4}$	1.07438	1.09569	1.09077	1.04044	0.92044	0.70152
$\sigma_{\varphi 5}$	1.07438	1.09137	1.08537	1.03847	0.92371	0.70323

In Table 1 we give the values of the successive approximations for the stress components in the form

$$\sigma_{rn} = \frac{1}{p}(\sigma_r^{\circ} + \sigma_r^1 + \ldots + \sigma_r^n), \qquad \sigma_{\varphi n} = \frac{1}{p}(\sigma_\varphi^{\circ} + \sigma_\varphi^1 + \ldots + \sigma_\varphi^n)$$

for the values $\varphi = 0$, $q = -1.6$, $\nu = 0.3$, which correspond to the values of the parameters in [1]. This allows us to compare the results.

Thus, the presence of nonhomogeneity implies the formation of shear stresses, although insignificant in magnitude. In addition a quantitative variation of the maximal values of σ_r by 9% and of σ_φ by 30%, is observed.

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ON THE MAGNETOELASTICITY OF THIN SHELLS AND PLATES

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We consider some problems of magnetoelastic oscillations of thin electrically conducting plates and shells situated in a stationary magnetic field. On this basis of the solutions, obtained by the method of asymptotic integration of the three-dimensional equations of magnetoelasticity, we formulate a hypothesis relative to the character of the variation of the electromagnetic field and of the elastic displacements along the thickness of the shell. This allows us to reduce the three-dimensional equations of magnetoelasticity to two-dimensional ones, which facilitates in an essential way the study of the magnetoelastic problems of thin bodies.

The problem of the investigation of magnetoelastic oscillations of electrically conducting shells in a magnetic field reduces to the simultaneous solution of the equations of magnetoelasticity in the domain occupied by the shell and the equations of the electrodynamics in the exterior of the shell. The equations of